Charles University in Prague
Faculty of Mathematics and Physics

BACHELOR THESIS

Karel Ha

Separation axioms

Department of Applied Mathematics

Supervisor of the bachelor thesis:	Prof. RNDr. Aleš Pultr, DrSc.
Study programme:	Computer Science
Specialization:	General Computer Science

Prague 2013

Acknowledgement

I wish to express my deepest gratitude to my advisor, professor Aleš Pultr, for his helpful suggestions, constructive criticism, excellent tutoring and, above all, for showing me how "pointless" topology can be.

I would also like to thank my loving and caring family, whose support in (not only) the most desperate times of my studies will always be appreciated.

My further thanks belong to my dear friend Filip Miller for all his fruitful comments concerning English language.

Last but not least, I would like to pay tribute to my patient mentors at the Charles University in Prague for guiding me through the thorny path of academic life and enabling me to truly develop as a person.

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Coll., the Copyright Act, as amended, in particular the fact that the Charles University in Prague has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 paragraph 1 of the Copyright Act.

In Prague, May 22, 2013 Karel Ha

Název práce: Oddělovací axiomy

Autor: Karel Ha

Katedra: Katedra aplikované matematiky

Vedoucí bakalářské práce: Prof. RNDr. Aleš Pultr, DrSc., Katedra aplikované matematiky

Abstrakt: Klasická (bodová) topologie se zabývá body a vztahy mezi nimi a určitými podmnožinami. Když odhlédneme od bodů a uvážíme pouze strukturu otevřených množin, získáme tzv. *frame* neboli úplný svaz splňující distributivní zákon $b \wedge \bigvee A = \bigvee\{b \wedge a \mid a \in A\}$. Ten je důležitým konceptem bezbodové topologie.

Bezbodový přístup (při téměř nepatrné ztrátě informací) nám poskytuje hlubší poznatky o topologii. Příkladem je studium oddělovacích axiomů. Tato práce je zaměřena na T_i-axiomy (pro $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$), tj. vlastnosti topologických prostorů zahrnující oddělování bodů od sebe, oddělování bodů od uzavřených množin a oddělování uzavřených množin samotných. V této práci probereme jejich bezbodové protějšky a způsoby, kterými na sobě závisí.

Klíčová slova: bezbodová topologie, oddělovací axiomy, locale, frame

Title: Separation axioms

Author: Karel Ha

Department: Department of Applied Mathematics

Supervisor: Prof. RNDr. Aleš Pultr, DrSc., Department of Applied Mathematics

Abstract: The classical (point-set) topology concerns points and relationships between points and subsets. Omitting points and considering only the structure of open sets leads to the notion of *frames*, that is, a complete lattice satisfying the distributive law $b \wedge \bigvee A = \bigvee\{b \wedge a \mid a \in A\}$, the crucial concept of point-free topology.

This pointless approach—while losing hardly any information—provides us with deeper insights on topology. One such example is the study of separation axioms. This thesis focuses on the T_i-axioms (for $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$): properties of topological spaces which regard the separation of points, points from closed sets, and closed sets from one another. In this text we discuss their point-free counterparts and how they relate to each other.

Keywords: pointless topology, point-free topology, separation axioms, locale, frame

Contents

Introduction **2**

I Preliminaries **3**
- I.1 Notation . . . 3
- I.2 Definitions . . . 3

II Subfitness **11**
- II.1 The T_1-axiom . . . 11
- II.2 Subfit locales . . . 11

III Weakly and strongly Hausdorff **13**
- III.1 The T_2-axiom . . . 13
- III.2 The Dowker-Strauss approach . . . 13
- III.3 Isbell's approach . . . 14

IV More Hausdorff type axioms **18**
- IV.1 Variants of (DS-Haus) . . . 18
  - IV.1.1 The axiom of Johnstone and Sun Shu-Hao . . . 18
  - IV.1.2 The axiom of Paseka and Šmarda . . . 18
  - IV.1.3 Another (stronger) variant . . . 18
- IV.2 Variants based on (semi)primeness . . . 19
  - IV.2.1 The axiom of Rosický . . . 19
- IV.3 Comparison . . . 19
- IV.4 Adding the subfitness . . . 20

V Regularity **21**
- V.1 The T_3-axiom . . . 21
- V.2 Regular locales . . . 22
- V.3 Strength of regularity . . . 22

VI Complete regularity **25**
- VI.1 The $T_{3\frac{1}{2}}$-axiom . . . 25
- VI.2 Completely regular locales . . . 25

VII Normality **27**
- VII.1 The T_4-axiom . . . 27
- VII.2 Normal locales . . . 27

Conclusion **29**

Bibliography **30**

Introduction

In classical topology, spaces are often specified by *separation axioms* of various strength. The stronger axiom we have, the more “geometrical” a space appears to be.

The best-known of them are the T_i-axioms ($i = 0, 1, 2, 3, 3\frac{1}{2}, 4$). They concern the separation of points, points from closed sets, and closed sets from each other. The role of points seems to be fundamental, and therefore, it may seem hard to find natural counterparts of the T-axioms in the point-free context. Nevertheless, most of them do have pointless equivalents. The aim of this thesis is to provide their summary and to show their mutual relationships.

Structure of thesis

The first chapter contains the necessary concepts of order and category theories, followed by the essentials of point-set and point-free topology.

In the subsequent chapter we discuss the *subfitness*—a weakened form of the T_1-axiom (the T_1-axiom itself does not seem to have a very natural counterpart). It should be noted that the subfitness is crucial for some relationships between other axioms. Besides, it makes good sense in classical topology as well, and moreover, plays a role in logic.

The *Hausdorff* axiom (T_2) does not have a unique direct counterpart. Unlike in the other cases, one has several non-equivalent alternatives. The two most standard ones are discussed in chapter III and in chapter IV we present a survey of several others.

Chapters V and VI are devoted to the *regularity* and to the *complete regularity*. We show how to exclude points from their definitions via “rather-below” relation $\prec$ and “completely-below” relation $\prec\!\!\!\prec$, and thus, how to create their point-free analogies.

The final chapter concerns the *normality*, the translation of which is perhaps the easiest one. We discuss the relation with the other axioms as well.

I. Preliminaries

I.1 Notation

- ♠ $\mathbb{N}$ resp. $\mathbb{Q}$ is the set of all natural resp. rational numbers.
- ♠ 2^X is the set of all subsets of a set X, that is $\{S \mid S \subseteq X\}$.
- ♠ $A \Subset B$ means that *set A is a finite subset of set B.*
- ♠ $\overline{S}$ is the closure of S in a given topology. It is defined by

$$\overline{S} = \bigcap_{\text{closed } A\colon\ A \supseteq S} A\,,$$

or equivalently,

$$\overline{S} = \{x \mid U \text{ is open and } U \ni x \Rightarrow U \cap S \neq \varnothing\}.$$

I.2 Definitions

Order theory

- ♠ A *preorder* is a binary relation $\sqsubseteq$ that is

  (reflexive) $a \sqsubseteq a$

  (transitive) $a \sqsubseteq b$ and $b \sqsubseteq c \Rightarrow a \sqsubseteq c$.

- ♠ A *pseudocomplement* of an element x is x^* such that $y \leq x^* \equiv x \wedge y = 0$.

Fact 1. $x \leq x^{**}$. (As the x, among others, meets with x^* in $x^* \wedge x = 0$.)

- ♠ A relation R *interpolates* if $R \subseteq R \circ R$.
- ♠ Monotone maps $l\colon A \to B$ and $r\colon B \to A$ are *Galois adjoints* (l is a *left adjoint* of r and r is a *right adjoint* of l) if for every $a \in A$, $b \in B$

$$l(a) \leq b \ \equiv\ a \leq r(b).$$

Remark I.2.1. Adjoints are unique:[1] $l(a) \leq b \ \equiv a \leq r(b) \ \equiv\ l'(a) \leq b$ leads to

$$\begin{aligned} l(a) \leq l'(a) \ &\equiv\ l'(a) \leq l'(a) \\ \text{and } l(a) \leq l(a) \ &\equiv\ l'(a) \leq l(a) \end{aligned}$$

and hence $l(a) = l'(a)$ for $a \in A$. Symmetrically for right adjoints.

Fact 2. $lr \leq id$ resp. $id \leq rl$. (Set $a := r(b)$ resp. $b := l(a)$.)

Fact 3. $lrl = l$. (Following from Fact 2 and the monotonicity of l we get $l \leq lrl$. On the other hand, $lr(b) \leq b$ with $b := l(a)$ gives us $lrl \leq l$.)

[1] if they exist

♠ f^* resp. f_* is the left resp. right adjoint of f.

♠ $\downarrow U = \bigcup_{u \in U}\{x \mid x \leq u\}$ and $\uparrow U = \bigcup_{u \in U}\{x \mid x \geq u\}$. Especially,

$$\downarrow a = \{x \mid x \leq a\} \text{ and } \uparrow a = \{x \mid x \geq a\}.$$

♠ A set U is a *down-set* resp. an *up-set* if $\downarrow U = U$ resp. $\uparrow U = U$.

♠ Let X be a partially ordered set. Then $\mathfrak{D}(X)$ is the set of all down-sets in X ordered by inclusion.

♠ We say that $p \neq 1$ is *prime* when

$$x_1 \wedge x_2 \leq p \implies x_1 \leq p \text{ or } x_2 \leq p,$$

and *semiprime* when

$$x_1 \wedge x_2 = 0 \implies x_1 \leq p \text{ or } x_2 \leq p.$$

Lemma I.2.2. *In a distributive lattice an element $p \neq 1$ is prime iff*

$$x_1 \wedge x_2 = p \implies x_1 = p \text{ or } x_2 = p.$$

Proof. $\Rightarrow$: If $x_1 \wedge x_2 = p$ then there is $x_i \leq p$ by primeness and since $x_j \geq x_1 \wedge x_2$ for $j \in \{1,2\}$ then also $x_i \geq p$.

$\Leftarrow$: We have $(x_1 \vee p) \wedge (x_2 \vee p) = (x_1 \wedge x_2) \vee p = p$. Hence, for some $i \in \{1,2\}$ we get $x_i \vee p = p$, that is, $x_i \leq p$. ■

♠ An *atom* resp. a *coatom* is an element $a > 0$ resp. $a < 1$ such that

$$0 < x \leq a \Rightarrow x = a$$
$$\text{resp. } a \leq x < 1 \Rightarrow x = a.$$

♠ For $a < b$ we say that a *is covered by* b and write $a \lessdot b$ if

$$a < x \leq b \Rightarrow x = b.$$

♠ An element a is said to be *covered* if it is covered by some b.

Category theory

♠ A *category* $\mathcal{C}$ is a collection consisting of

- class $ob(\mathcal{C})$ of *objects*
- class $morph(\mathcal{C})$ of *morphisms* between objects[2]

further satisfying the axioms of

- *(identity)* For every $X \in ob(\mathcal{C})$ there is a morphism 1_X such that for any $f\colon A \to B$ we have $1_B \cdot f = f = f \cdot 1_A$.

[2]a morphism f between objects A and B is indicated by $f\colon A \to B$

– *(asociativity)* Whenever $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ then also $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

♠ The *opposite category* $\mathcal{C}^{op}$ of a category $\mathcal{C}$ has the same objects $ob(\mathcal{C})$ and reverses the composition. Thus a morphism $A \to B$ goes now from B to A.

♠ Let objects of a category $\mathcal{C}$ be structured sets, their morphisms be maps (respecting the structure in that or other way) and the composition of morphisms be same as the composition of maps. Then $\mathcal{C}$ is called *concrete*.

♠ An *epimorphism* is a right-cancellative morphism e, that is,

$$f_1 \cdot e = f_2 \cdot e \implies f_1 = f_2.$$

♠ A *monomorphism* is a left-cancellative morphism m:

$$m \cdot f_1 = m \cdot f_2 \implies f_1 = f_2.$$

We also refer to these by saying that e *is epic* and m *is monic*.

Fact 4. In a concrete category every morphism onto resp. one-one is epic resp. monic.

(Else there would be x with $f_1(x) \neq f_2(x)$. From surjectivity, the existence of z such that $e(z) = x$ would lead to $f_1 \cdot e(z) \neq f_2 \cdot e(z)$; and if m is one-one then $m \cdot f_1(x) \neq m \cdot f_2(x)$.)

♠ An *isomorphism* is a morphism $f\colon A \to B$ that has an *inverse morphism* $\overline{f}\colon B \to A$ such that

$$\overline{f} \cdot f = 1_A \text{ and } f \cdot \overline{f} = 1_B.$$

♠ A *product* of a system $(A_i)_{i\in J}$ is a system of *projections*

$$\left(p_i\colon \prod_{j\in J} A_j \to A_i\right)_{i\in J}$$

with the property that for an arbitrary system $(f_i\colon X \to A_i)_{i\in J}$ there is a unique solution $f\colon X \to \prod_{j\in J} A_j$ to equations

$$p_i \cdot f = f_i \text{ for } i \in J.$$

♠ Similarly, a *coproduct* is a system of *injections*

$$\left(\iota_i\colon A_i \to \coprod_{j\in J} A_j\right)_{i\in J}$$

such that any $(g_i\colon A_i \to X)_{i\in J}$ has a single solution $g\colon \coprod_{j\in J} A_j \to X$ to equations

$$g \cdot \iota_i = g_i \text{ for } i \in J.$$

Note that this is a product in the opposite category.

♠ The *diagonal* is the (only) morphism $\Delta\colon A \to \prod_{i\in J} A$ solving the system $p_i \cdot \Delta = 1_A$ for $i \in J$. Dually, the *codiagonal* is the solution $\nabla\colon \coprod_{i\in J} A \to A$ to the system $\nabla \cdot \iota_i = 1_A$ for $i \in J$.

Point-set topology

♠ A *topology* on a set X is a system $\tau \subseteq 2^X$ satisfying the axioms

(o1) $\varnothing, X \in \tau$

(o2) $\{U_i \mid i \in J\} \subseteq \tau \;\Rightarrow\; \bigcup_{i\in J} U_i \in \tau$

(o3) $U_1, U_2 \in \tau \;\Rightarrow\; U_1 \cap U_2 \in \tau$.

♠ The $U \in \tau$ are referred to as *open sets* and (X, τ) as a *topological space*.

♠ We often abbreviate (X, τ) to X; then τ is usually indicated by $\Omega(X)$.

♠ The complements of open sets are referred to as *closed sets*.

♠ A *basis* of a topology τ is $\mathcal{B} \subseteq \tau$ such that for every open U we can write $U = \bigcup\{V \in \mathcal{B} \mid V \subseteq U\}$.

♠ $\mathcal{S} \subseteq \tau$ is a *subbasis* of a topology τ if $\{\bigcap \mathcal{F} \mid \mathcal{F} \Subset \mathcal{S}\}$ is a basis of τ.

Example 1. $\{\langle 0, a\langle,\ \rangle a, 1\rangle \mid 0 < a < 1\}$ is a subbasis of the standard topology of the interval $\langle 0, 1\rangle$. This topological space is denoted by $\mathbb{I}$.

♠ A function $f\colon X \to Y$ between topological spaces X and Y is *continuous* if

$$U \in \Omega(Y) \implies f^{-1}[U] \in \Omega(X).$$

Since the preimage function preserves unions and finite intersections, it suffices to require that $f^{-1}[U]$ be open for all *subbasic* U.

Example 2. The topological space $\left(\prod_{i\in J} X_i, \mathcal{S}\right)$ with the subbasis

$$\mathcal{S} = \left\{p_i^{-1}[U] \mid i \in J,\ U \in \tau_i\right\}$$

is the product of topological spaces $(X_i)_{i\in J}$; more precisely, the projections

$$\left(p_i\colon \left(\prod_{i\in J} X_i, \mathcal{S}\right) \to (X_i, \tau_i)\right)_{i\in J}$$

constitute the product in the category **Top** (of topological spaces and continuous functions).

Point-free topology

♠ A *frame* is a complete lattice L satisfying the distributivity law

$$b \wedge \bigvee A = \bigvee\{b \wedge a \mid a \in A\}$$

for every $A \subseteq L$ and $b \in L$.

♠ For a topological space X the set $\Omega(X)$ ordered by inclusion is a frame: by axioms **(o2)** and **(o3)** (see "Point-set topology") operations $\wedge$ resp. $\bigvee$ coincide with $\cap$ resp. $\bigcup$.

- ♠ A *frame homomorphism* between frames M and L is a map $h\colon M \to L$ preserving all joins (in particular, $h(0) = 0$) and all finite meets (especially, $h(1) = 1$).
- ♠ The category of frames and frame homomorphisms will be denoted by **Frm**.
- ♠ The objects of the opposite category **Loc** are called *locales*. The morphisms of **Loc** can be represented as follows:
  - A *localic map* between locales L and M is a mapping $f\colon L \to M$ with left Galois adjoints $f^*\colon M \to L$ such that f^* is a frame homomorphism. Thus, localic maps are infima-preserving maps such that their left adjoints preserve finite meets.
- ♠ A *sublocale* of a locale L is $S \subseteq L$ satisfying

  (S1) $M \subseteq S \Longrightarrow \bigwedge M \in S$

  (S2) $x \in L$ and $s \in S \Longrightarrow x \to s \in S$.

Fact 5. The set-theoretic image $f[L]$ under a localic map $f\colon L \to M$ is a sublocale of M.

- ♠ Each $\uparrow a$ is a sublocale. These sublocales are referred to as *closed sublocales*.

Remark I.2.3 (Sublocale homomorphisms). There is an alternative representation of sublocales:

- ♠ A *sublocale* of a locale L is a frame homomorphism $h\colon L \to M$ that is onto. For sublocales $h\colon L \to M$, $h'\colon L \to M'$ we have the preorder:

  $$h \sqsubseteq h'$$

  if there is a frame homomorphism $\alpha\colon M' \to M$ such that $\alpha h' = h$.
- ♠ Thus, h and h' represent the same sublocale iff this α is also an isomorphism.
- ♠ In this representation *closed sublocales* are frame homomorphisms which are of the form

  $$\check{a} = (x \mapsto a \vee x)\colon L \to \uparrow a$$

  for $a \in L$.

Lemma I.2.4. *The embedding $j = (x \mapsto x)\colon \uparrow a \xrightarrow{\subseteq} L$ is the right adjoint to $\check{a}$.*

Proof. For $x \in L$ and $y \in \uparrow a$ we will show that $x \vee a = \check{a}(x) \leq y$ iff $x \leq j(y) = y$.
$\Rightarrow$: Immediately from $x \leq x \vee a$.
$\Leftarrow$: Following from $y \geq x$ and $y \geq a$. ■

- ♠ The *closure* of a sublocale $h\colon L \to M$ is the $\sqsubseteq$-least closed sublocale f such that $f \sqsupseteq h$.

Proposition I.2.5. *For a locale L and any of its sublocales $h\colon L \to M$ let*

$$c := \bigvee\{x \mid h(x) = 0\}.$$

Then $\check{c}\colon L \to \uparrow c$ is the closure of h.

Proof. Firstly, $\check{c} \sqsupseteq h$: let us find $\gamma\colon \uparrow c \to M$ with $\gamma\check{c} = h$. For every $y \in \uparrow c$ take $x \in L$ such that $y = x \vee c$ and set $\gamma(y) := h(x)$. It is a correct definition: since $h(c) = \bigvee\{h(x) \mid h(x) = 0\} = 0$, if $x_1 \vee c = x_2 \vee c$ then

$$h(x_1) = h(x_1) \vee 0 = h(x_1 \vee c) = h(x_2 \vee c) = h(x_2) \vee 0 = h(x_2).$$

Furthermore, γ is a frame homomorphism as $\check{c}$ and $\gamma\check{c} = h$ are homomorphisms and $\check{c}$ is onto.

Now suppose $\check{a} \sqsupseteq h$ for some $a \in L$. That is, there exists a frame homomorphism $\alpha\colon \uparrow a \to M$ with $\alpha\check{a} = h$. We have

$$0 = h(0) = \alpha\check{a}(0) = \alpha(a \vee 0) = \alpha(a \vee a) = \alpha\check{a}(a) = h(a).$$

From the definition of c we see that $a \leq c$. Set $\psi := (x \mapsto c \vee x)\colon \uparrow a \to \uparrow c$. We have $\psi\check{a}(x) = c \vee a \vee x = c \vee x = \check{c}(x)$, and consequently, $\check{a} \sqsupseteq \check{c}$. ■

With the same notation as above, we get

Corollary I.2.6. *A sublocale h is closed iff there exists a map $\varphi\colon M \to \uparrow c$ with $\varphi h = \check{c}$.*

Proof. Since h and $\varphi h = \check{c}$ are frame homomorphisms and h is onto, φ is also a homomorphism. Moreover, both h and $\check{c}$ are onto and hence epic, and thus,

$$h = \gamma\check{c} = \gamma\varphi h \Longrightarrow \gamma\varphi = id,$$
$$\check{c} = \varphi h = \varphi\gamma\check{c} \Longrightarrow \varphi\gamma = id.$$

Therefore, φ and γ are mutually inverse isomorphisms. ■

On coproducts in Frm

In the category of frames we have a coproduct

$$\iota_i : L_i \to L_1 \oplus L_2,\ i = 1, 2$$

(there is a general coproduct but we will need a coproduct of two objects only). It can be constructed as follows.

- ♠ A down-set $U \subseteq L_1 \times L_2$ is said to be *saturated* if for every system $(x_i)_{i\in J}$ in L_1 resp. L_2 and every y in L_2 resp. L_1 we have

$$\{(x_i, y) \mid i \in J\} \subseteq U \ \Rightarrow\ (\bigvee x_i, y) \in U$$
$$\text{and } \{(y, x_i) \mid i \in J\} \subseteq U \ \Rightarrow\ (y, \bigvee x_i) \in U.$$

- ♠ The definition above concerns also the void J and hence for each saturated down-set U we get

$$U \supseteq \mathfrak{n} := \{(x, y) \mid x = 0 \text{ or } y = 0\}.$$

Since $\mathfrak{n}$ itself is obviously saturated, it is *the least saturated down-set*.

- ♠ $L_1 \oplus L_2$ will be the set of all saturated elements in $\mathfrak{D}(L_1 \times L_2)$. In other words, $L_1 \oplus L_2$ is the frame of all saturated down-sets of $L_1 \times L_2$ ordered by the inclusion.

Proposition I.2.7. *For $(a, b) \in L_1 \times L_2$ the down-set*

$$\downarrow (a, b) \cup \mathfrak{n}$$

is saturated.

Proof. Let $(x_i, y) \in \downarrow (a, b) \cup \mathfrak{n}$ for $i \in J$.

Case $y = 0$: evidently, $(\bigvee_{i \in J} x_i, y) \in \mathfrak{n}$.

Case $y \neq 0$ and $\bigvee x_i = 0$: again $(\bigvee_{i \in J} x_i, y) \in \mathfrak{n}$.

Case $y \neq 0$ and $\bigvee x_i \neq 0$: Then $x_t \neq 0$ for some $t \in J$. Thus, $(x_t, y) \in \downarrow (a, b)$, and hence, $(x_i, y) \leq (a, b)$ for all $i \in J$. Finally, $(\bigvee_{i \in J} x_i, y) \in \downarrow (a, b)$.

Symmetrically for $(x, \bigvee_{i \in J} y_i)$. ■

- ♠ This element of $L_1 \oplus L_2$ is denoted by

$$a \oplus b$$

and we have the coproduct injections $\iota_i : L_i \to L_1 \oplus L_2$ defined by

$$\iota_1(a) := a \oplus 1, \ \iota_2(b) := 1 \oplus b.$$

We will not prove that frame $L_1 \oplus L_2$ constitutes a coproduct. For full details consult Chapter IV of Picado and Pultr [5].

Observation. We have
$$a \oplus b = \iota_1(a) \wedge \iota_2(b).$$

- ♠ Note that for all saturated U we have

$$U = \bigcup \{\downarrow (a, b) \mid (a, b) \in U\} = \bigcup \{a \oplus b \mid (a, b) \in U\}.$$

As the set-theoretic union of down-sets is also a down-set, it coincides with their join. Therefore, we may write

$$U = \bigvee \{a \oplus b \mid (a, b) \in U\} = \bigvee \{a \oplus b \mid a \oplus b \subseteq U\}$$

and thus *the elements $a \oplus b$ generate $L_1 \oplus L_2$.*

♠ For the codiagonal $\nabla\colon L \oplus L \to L$ in **Frm** we have

$$\nabla(U) = \nabla\left(\bigvee \{a \oplus b \mid (a,b) \in U\}\right) = \bigvee \{\nabla(a \oplus b) \mid (a,b) \in U\}\,.$$

Further,

$$\nabla(a \oplus b) = \nabla(\iota_1(a) \wedge \iota_2(b)) = \nabla(\iota_1(a)) \wedge \nabla(\iota_2(b)) = a \wedge b$$

since $\nabla \cdot \iota_i = id$ for $i = 1, 2$. In conclusion,

$$\boxed{\nabla(U) = \bigvee \{a \wedge b \mid (a,b) \in U\} = \bigvee \{x \mid (x,x) \in U\}}$$

because $(a,b) \in U \Rightarrow (a \wedge b, a \wedge b) \in U$ for any down-set U.

♠ Note that the codiagonal is epic. As $\nabla \cdot \iota_1 = id$, we have

$$f_1 \cdot \nabla = f_2 \cdot \nabla \implies f_1 \cdot \nabla \cdot \iota_1 = f_2 \cdot \nabla \cdot \iota_1 \implies f_1 = f_2$$

♠ The diagonal $\Delta\colon L \to L \oplus L$ in the (opposite) category **Loc** is the right adjoint of ∇. Thus, we require that

$$\begin{aligned} U \subseteq \Delta(a) \;\equiv\; \nabla(U) \le a \;&\equiv\; \bigvee \{u_1 \wedge u_2 \mid (u_1,u_2) \in U\} \le a \\ &\equiv\; \forall (u_1,u_2) \in U\colon\; u_1 \wedge u_2 \le a, \end{aligned}$$

which—after setting $U := \Delta(a)$—produces the final formula for the diagonal

$$\boxed{\Delta(a) = \{(u_1,u_2) \mid u_1 \wedge u_2 \le a\}\,.}$$

Convention

In the whole text we we will assume all topological spaces to be T_0-spaces. Thus, every topological space (X,τ) satisfies the condition

for any $x \neq y$ from X there is an open set $U \in \tau$ such that $x \in U \not\ni y$ or $y \in U \not\ni x$.

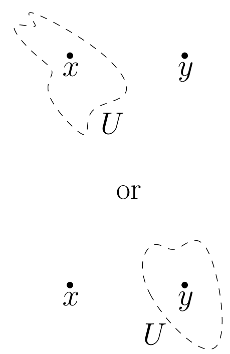

Figure A: T_0 property

II. Subfitness

II.1 The T_1-axiom

Definition (T_1)**.** A topological space (X, τ) is said to be a T_1-*space* (or to satisfy the T_1-*axiom*) if the following holds:

for any $x \neq y$ from X there exists an open set $U \in \tau$ such that $x \in U \not\ni y$.

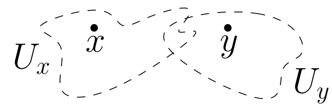

Figure B: T_1 property
(U_x, U_y are not necessarily disjoint)

The following is a standard and useful characterization of this property.

Fact 6. A topological space X is T_1 iff $\overline{\{x\}} = \{x\}$ for any $x \in X$.

(Of course: if $y \notin \{x\}$, we obtain an open U such that $y \in U \not\ni x$ and thus $y \notin \overline{\{x\}}$. Conversely, with open $U := X \setminus \{y\}$ we can easily check (T_1).)

Corollary II.1.1. *A space is T_1 iff every of its finite subsets is closed.*

There are exact counterparts of this axiom in the point-free context (see, for instance, [1]). They have not found much use so far, though. Instead, one introduces a weaker and very useful condition: *the subfitness*.

II.2 Subfit locales

Definition (Sfit)**.** A locale L is said to be *subfit* if

$$a \not\leq b \qquad \Rightarrow \qquad \exists c, \quad a \vee c = 1 \neq b \vee c.$$

Theorem II.2.1. *Every T_1-space is subfit.*

Proof. For each T_1-space X the corresponding locale $\Omega(X)$ is subfit. Indeed: for open $A \not\subseteq B$ there exists an $x \in A \setminus B$. By Fact 6 we have an open $X \setminus \{x\}$. Since $x \in A$ and $x \notin B$, we conclude with $A \cup (X \setminus \{x\}) = X \neq B \cup (X \setminus \{x\})$. ■

The subfit topological spaces are characterized by

Proposition II.2.2 (Isbell, Simmons). *For a topological space X the locale $\Omega(X)$ is subfit iff*

$$\forall U \in \Omega(X) \forall x \in U \exists y \in \overline{\{x\}}: \quad \overline{\{y\}} \subseteq U$$

Proof. $\Rightarrow$: Choose an $x \in U$. Thus, we have $U \nsubseteq X \setminus \overline{\{x\}}$ and by subfitness an open V with $U \cup V = X \neq (X \setminus \overline{\{x\}}) \cup V$. Then there exists an element $y \notin V$ with $y \notin X \setminus \overline{\{x\}}$, in other words, $y \in \overline{\{x\}}$. Furthermore, $V \cap \overline{\{y\}} = \varnothing$ (for $z \in V \not\ni y$ means $z \notin \overline{\{y\}}$), and as $U \cup V = X$, we finish with $\overline{\{y\}} \subseteq U$.

$\Leftarrow$: Suppose $U \nsubseteq V$ and take $x \in U \setminus V$. For the similar reason as above $V \cap \overline{\{x\}} = \varnothing$. Let us have y from the premise: $y \in \overline{\{x\}}$ implies $y \notin V$, thus, leading to $(X \setminus \overline{\{y\}}) \cup V \neq X$; and since $\overline{\{y\}} \subseteq U$ then also $(X \setminus \overline{\{y\}}) \cup U = X$. ■

Here is an example of a subfit non-T_1-space:

Example 3. Let us have $X = \mathbb{N} \cup \{\infty\}$ and $\theta = \{F \subseteq X \mid X \setminus F \Subset \mathbb{N}\} \cup \{\varnothing\}$. That is, $\mathcal{X} := (X, \theta)$ is the topological space where closed sets consist of finite sets of natural numbers and of the whole space X.

It is not T_1 for the one-point set $\{\infty\}$ is not closed (see Fact 6).

On the other hand, any open set satisfies the condition from Proposition II.2.2: the empty set trivially; since a finite $\{x\}$ for $x \in \mathbb{N}$ is closed itself, we are allowed to put $y := x$; and if $x = \infty$ then $\overline{\{\infty\}} = X$, hence we can choose any $y \neq \infty$ from U.

III. Weakly and strongly Hausdorff

In this chapter we will consider two point-free analogies of the T_2-axiom: a separation not quite easy to imitate in frames.

III.1 The T_2-axiom

Definition (T_2)**.** A topological space (X, τ) is called *Hausdorff* (also equally a *T_2-space*) if

for any $x \neq y$ from X there are disjoint U and V with $U \ni x$ and $V \ni y$.

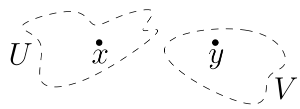

Figure C: T_2 property

Remark III.1.1. Trivially, every T_2-space is T_1.

III.2 The Dowker-Strauss approach

In 1972 Dowker and Strauss [1] introduced the following condition:

Definition. A locale L satisfies the S_2'-axiom if

$$(S_2') \qquad a, b \neq 1 \text{ and } a \vee b = 1 \quad \Rightarrow \quad \exists u \not\leq a, v \not\leq b, \quad u \wedge v = 0.$$

Its relationship to the Hausdorff property is seen from following

Theorem III.2.1. *For the $\Omega(X)$ of any topological space X we have*

$$(T_2) \;\equiv\; (S_2')\&(Sfit).$$

Lemma III.2.2. *The $\Omega(X)$ of a Hausdorff X satisfies S_2'.*

Proof. Open sets $A, B \neq X$ with $A \cup B = X$ contain some $x \in X \setminus A = B \setminus A$ and some $y \in X \setminus B = A \setminus B$. Particularly, we observe $x \neq y$ and, on top of that, receive the disjoint U and V from (T_2). Also, $U \not\subseteq A$ (because of x) and $V \not\subseteq B$ (because of y). ∎

Lemma III.2.3. $(S_2')\&(\mathit{Sfit}) \Rightarrow (T_1)$.

Proof. Using Fact 6 on page 11 we will show that $\{x\} = \overline{\{x\}}$.

Assume, for the sake of contradiction, that there is $z \in \overline{\{x\}} \setminus \{x\}$. Since we still suppose (T_0), we have $x \notin \overline{\{z\}}$. Recall Proposition II.2.2 on page 12 and consider $y \in \overline{\{x\}}$ with $\overline{\{y\}} \subseteq X \setminus \overline{\{z\}}$, that is, $\overline{\{y\}} \cap \overline{\{z\}} = \varnothing$.

For open $A := X \setminus \overline{\{y\}}$ and $B := X \setminus \overline{\{z\}}$ we have $A \cup B = X$, and hence, also the U, V from (S_2'). Thus, there must be $y' \in \overline{\{y\}} \cap U$, and consequently, $y \in U$ (else $y' \in U \not\ni y$ would lead to $y' \notin \overline{\{y\}}$). Likewise, $z \in V$. However, U and V cannot be disjoint: as $y \in \overline{\{x\}}$ resp. $z \in \overline{\{x\}}$ implies $x \in U$ resp. $x \in V$. ■

Lemma III.2.4. $(S_2')\&(T_1) \Rightarrow (T_2)$.

Proof. For open $A := X \setminus \{x\}$ and $B := X \setminus \{y\}$ take the disjoint $U, V \in \Omega(X)$ from (S_2'). Thus, $U \not\subseteq A$ resp. $V \not\subseteq B$ translates to $U \ni x$ resp. $V \ni y$. ■

Proof of **III.2.1**.

$\Rightarrow$: By III.1.1 and II.2.1 (on page 11) we get the subfitness and using III.2.2 also the axiom (S_2').

$\Leftarrow$: Apply lemmata III.2.3 and III.2.4. ■

The (S_2') axiom is often replaced by a somewhat stronger condition:

Definition (DS-Haus)**.** A locale L is said to be *weakly Hausdorff* (also *DS-Hausdorff* as in *Dowker-Straus-Hausdorff*) if

$$a \vee b \notin \{a, b\} \qquad \Rightarrow \qquad \exists u \not\leq a, v \not\leq b, \quad u \wedge v = 0.$$

III.3 Isbell's approach

Here is a characterization of the classical (T_2):

Proposition III.3.1. *A topological space X is Hausdorff space if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in the product $X \times X$.*

Proof. $\Rightarrow$: Any element $(x, y) \notin \Delta$, that means $x \neq y$, is separable from Δ: namely, by the open $p_1^{-1}[U] \cap p_2^{-1}[V]$ with disjoint U and V from the definition of (T_2). Hence, $(x, y) \notin \overline{\Delta}$ concludes to $\Delta = \overline{\Delta}$, which is closed.

$\Leftarrow$: Choose $x \neq y$. In other words, $(x, y) \notin \Delta$. Then (x, y) has an open basic neighborhood $U_1 \times U_2$ disjoint from Δ. That is, $U_1 \cap U_2 = \varnothing$. ■

This was used by Isbell [2] for his point-free counterpart of the T_2-axiom: Hausdorff locales are defined as those in which the codiagonal (see page 10)

$$\nabla \colon L \oplus L \to L$$

is a closed sublocale (see page 7). Since $\nabla(U) \leq 0 \;\equiv\; U \subseteq \Delta(0)$, the closure of the codiagonal (recall I.2.5) is $\check{d_L} = (U \mapsto U \vee d_L)$ where

$$d_L = \bigvee\{U \mid \nabla(U) = 0\} = \bigvee\{U \mid U \subseteq \Delta(0)\} = \Delta(0).$$

Following from Corollary I.2.6 the condition proposed by Isbell reduces to

Definition (I-Haus)**.** A frame L is *strongly Hausdorff* (or *I-Hausdorff*: short for *Isbell-Hausdorff*) if there is a mapping $\alpha\colon L \to {\uparrow} d_L$ such that

$$\alpha\nabla = (U \mapsto U \vee d_L).$$

Note that the isomorphism α has to be the restriction of Δ to $L \to {\uparrow} d_L$. We have $\alpha^{-1}\check{d_L} = \nabla$, and hence,

$$\Delta = \nabla_* = (\alpha^{-1}\check{d_L})_* = (\check{d_L})_*\alpha = j\alpha$$

because the embedding $j\colon {\uparrow} d_L \xrightarrow{\subseteq} L \oplus L$ is the right adjoint to $\check{d_L}$ (see I.2.4).

Lemma III.3.2. *In an I-Hausdorff locale L it holds that $\Delta[L] = {\uparrow} d_L$.*

Proof. As α is the restriction of Δ and also an isomorphism, we get $\Delta[L] = \alpha[L] = {\uparrow} d_L$. ■

Theorem III.3.3. *(I-Haus) implies (DS-Haus).*

Lemma III.3.4. *In case of a strongly Hausdorff locale L, we have*

$$\Delta\nabla(U) = U$$

for any saturated $U \supseteq d_L$.

Proof. By Lemma III.3.2 every saturated $U \in {\uparrow} d_L = \Delta[L]$ is an image $\Delta(a)$ for some $a \in L$. Let us write $\delta(U)$ for this a. In other words, $\Delta\delta(U) = U$.

The frame codiagonal is an epimorphism and hence (as we have $\nabla\Delta\nabla = \nabla$ by Fact 3 on page 3) we also get $\nabla\Delta = id$.

Joining these two observations:

$$\nabla(U) = \nabla(\Delta\delta(U)) = (\nabla\Delta)\delta(U) = \delta(U),$$

which produces the desired $U = \Delta\delta(U) = \Delta\nabla(U)$. ■

Proposition III.3.5. *A locale L is I-Hausdorff iff one has the implication*

$$(a \wedge b, a \wedge b) \in U \implies (a, b) \in U \tag{3.a}$$

for all saturated $U \supseteq d_L$.

Proof. ⇒: Whenever $(a \wedge b, a \wedge b) \in U$, we see from the formula for ∇ that

$$a \wedge b \leq \bigvee \{x \mid (x, x) \in U\} = \nabla(U).$$

Having in mind the formula for Δ, immediately $(a, b) \in \Delta(\nabla(U)) = U$ (the equality by Lemma III.3.4).

$\Leftarrow$: Let the condition hold and let $a_i \in L$ for $i \in J$. Since for any saturated down-set $U \supseteq d_L$ we have $(a_i, a_i) \in U \Rightarrow (a_i \wedge a_j, a_i \wedge a_j) \in U$, we also obtain the implication

$$(a_i, a_i) \in U \Longrightarrow \{(a_i, a_j) \mid j \in J\} \subseteq U \tag{3.b}$$

We will prove that $\alpha := (a \mapsto (a \oplus a) \vee d_L)$ satisfies the definition. Using

$$\bigvee_{i\in J} a_i \oplus \bigvee_{j\in J} a_j = \iota_1\left(\bigvee_{i\in J} a_i\right) \wedge \iota_2\left(\bigvee_{j\in J} a_j\right) = \bigvee_{i,j\in J} (\iota_1(a_i) \wedge \iota_2(a_j)) = \bigvee_{i,j\in J} (a_i \oplus a_j)$$

(the latter equality by double application of frame distributivity), we see that α preserves suprema:

$$\alpha\left(\bigvee_{i\in J} a_i\right) = \left(\bigvee_{i,j\in J} (a_i \oplus a_j)\right) \vee d_L = \bigvee_{i\in J} ((a_i \oplus a_i) \vee d_L) = \bigvee_{i\in J} \alpha(a_i)$$

(by (3.b) the second and the third expression have the identical upper bounds in $L \oplus L$). Thus, both α and ∇ preserve joins and so does $\alpha\nabla$.

Furthermore, we have

$$\alpha\nabla(a \oplus b) = \alpha(a \wedge b) = ((a \wedge b) \oplus (a \wedge b)) \vee d_L = (a \oplus b) \vee d_L,$$

the last equality stipulated by the assumed implication (3.a).

In conclusion,

$$\begin{aligned}\alpha\nabla(U) &= \alpha\nabla\left(\bigvee\{a \oplus b \mid (a,b) \in U\}\right) = \bigvee\{\alpha\nabla(a \oplus b) \mid (a,b) \in U\} \\ &= \bigvee\{(a \oplus b) \vee d_L \mid (a,b) \in U\} = \bigvee\{a \oplus b \mid (a,b) \in U\} \vee d_L = U \vee d_L,\end{aligned}$$

as the elements $a \oplus b$ generate $L \oplus L$ (see page 9). ■

Lemma III.3.6. *The down-set*

$$U = {\downarrow}(a, a \wedge b) \cup {\downarrow}(a \wedge b, b) \cup \mathfrak{n}$$

is saturated in $L \times L$ for arbitrary $a, b \in L$.

Proof. Recall the saturatedness on page 8. First of all, let us have $(x_i, y) \in U$ for $i \in J$.

Case $y = 0$: obviously, $(\bigvee_{i\in J} x_i, y) \in \mathfrak{n}$.

Case $y \neq 0$ and $y \leq a \wedge b$: then it must be that $x_i \leq a$ in any case. Thus, $(\bigvee_{i\in J} x_i, y) \in {\downarrow}(a, a \wedge b)$.

Case $y \neq 0$ and $y \not\leq a \wedge b$: we have both $y \leq b$ and $x_i \leq a \wedge b$. Hence, again $(\bigvee_{i\in J} x_i, y) \in {\downarrow}(a \wedge b, b)$.

The $(x, \bigvee_{i\in J} y_i)$ by symmetry. ■

Now we can prove the theorem. In fact, we are going to imply a stronger version of the (DS-Haus):

$$a \vee b \notin \{a, b\} \qquad \Rightarrow \qquad \exists u \not\leq b, v \not\leq a : \quad u \wedge v = 0, \quad \boxed{(u, v) \leq (a, b)}$$

Proof of **III.3.3**. For a contradiction: let there be an I-Hausdorff L, which is not weakly Hausdorff. That means, we have a, b with $a \not\leq b$, $b \not\leq a$ and such that

$$u \wedge v = 0 \quad \& \quad (u, v) \leq (a, b) \qquad \Longrightarrow \qquad u \leq b \ \text{ or } \ v \leq a.$$

Especially, for the down-set U taken from III.3.6 we have

$$d_L \cap (a \oplus b) \subseteq U. \tag{3.c}$$

Following from III.3.5 the saturated $((a \wedge b) \oplus (a \wedge b)) \vee d_L \supseteq d_L$ contains (a, b) as it trivially contains $(a \wedge b, a \wedge b)$. Hence,

$$\begin{aligned}(a, b) &\in (a \oplus b) \ \cap \ (((a \wedge b) \oplus (a \wedge b)) \vee d_L) \\ &= ((a \wedge b) \oplus (a \wedge b)) \ \vee \ ((a \oplus b) \cap d_L) \\ &\subseteq ((a \wedge b) \oplus (a \wedge b)) \ \vee \ U \\ &= U.\end{aligned}$$

The first equality using distributivity and the fact that $(a \wedge b) \oplus (a \wedge b) \subseteq a \oplus b$; the inclusion by (3.c); the final equality from $(a \wedge b) \oplus (a \wedge b) \subseteq U$.

This leads to the contradiction: $(a, b) \notin \mathfrak{n}$ since neither a nor b may equal 0 (as $a \not\leq b$, $b \not\leq a$). Thus, the only options available are either $a \leq a \wedge b$ or $b \leq a \wedge b$. However, those stipulate $a \leq b$ or $b \leq a$ respectively, a contradiction. ■

IV. More Hausdorff type axioms

There are other (T_2)-type axioms for frames. The aim of this chapter is to give a confrontation of the Dowker-Strauss definition with axioms proposed by other pointless topologists.

IV.1 Variants of (DS-Haus)

IV.1.1 The axiom of Johnstone and Sun Shu-Hao

Johnstone and Sun Shu-Hao [3] suggest the axiom

$$(T_2') \qquad 1 \neq a \not\leq b \quad \Rightarrow \quad \exists u \not\leq a, v \not\leq b, \quad u \wedge v = 0.$$

Let us also introduce its weaker version

$$(S_<) \qquad 1 \neq a > b \quad \Rightarrow \quad \exists u \not\leq a, v \not\leq b, \quad u \wedge v = 0.$$

The axiom (T_2') is stronger than the axiom of Dowker and Papert-Strauss:

Proposition IV.1.1. $(T_2') \;\equiv\; (\mathit{DS\text{-}Haus})\&(S_<)$.

Proof. $\Rightarrow$: From $a \not\leq b \;\&\; b \not\leq a \Rightarrow a \neq 1$ we infer (DS-Haus). From $a > b \Rightarrow a \not\leq b$ we get $(S_<)$.

$\Leftarrow$: If $a \not\leq b$ then either $a > b$ or $a \not\geq b$. For the former case we apply $(S_<)$; for the latter one (DS-Haus). ■

IV.1.2 The axiom of Paseka and Šmarda

Paseka and Šmarda [4] propose a stronger version of (T_2'):

$$(\overline{T}_2) \qquad 1 \neq a \not\leq b \quad \Rightarrow \quad \exists u \not\leq a, v \not\leq b, \boxed{v \leq a}, \quad u \wedge v = 0.$$

Once more, we have its weaker version

$$(\overline{T}_<) \qquad 1 \neq a > b \quad \Rightarrow \quad \exists u \not\leq a, v \not\leq b, v \leq a, \quad u \wedge v = 0.$$

However, the situation is (or appears to be) different from Proposition IV.1.1:

Proposition IV.1.2. $(\overline{T}_2) \;\equiv\; (\overline{T}_<)$.

Proof. $\Rightarrow$: Because $a > b \Rightarrow a \not\leq b$.

$\Leftarrow$: If $a \not\leq b$ then $a \wedge b < a$. Applying $(\overline{T}_<)$ to $1 \neq a > a \wedge b$, we get $u \not\leq a, v \not\leq a \wedge b, v \leq a$ and $u \wedge v = 0$. Furthermore, $v \not\leq b$ (since otherwise $v \leq a \wedge b$). ■

IV.1.3 Another (stronger) variant

Here is another version of (DS-Haus) that is stronger:

$$(\overline{S}_2) \qquad a \vee b \notin \{a, b\} \quad \Rightarrow \quad \exists u \not\leq a, v \not\leq b, \boxed{u \leq b, v \leq a}, \quad u \wedge v = 0.$$

It was used in the proof of III.3.3 (see page 16).

IV.2 Variants based on (semi)primeness

IV.2.1 The axiom of Rosický

Rosický [6] has the following axiom:

$$(S) \qquad \text{Every semiprime element is a coatom.}$$

Two of its weaker variants are

$$(S_w) \qquad \text{Every semiprime element is covered.}$$

and

$$(S_{ww}) \qquad \text{Every semiprime element is prime.}$$

Proposition IV.2.1. *In a frame we have* $(S) \Rightarrow (S_w) \Rightarrow (S_{ww})$.

Proof. $(S) \Rightarrow (S_w)$: Coatoms are covered by 1.

$(S_w) \Rightarrow (S_{ww})$: We will show the equivalent condition from Lemma I.2.2 on page 4. For a semiprime $p = x_1 \wedge x_2$ take $b := \bigwedge\{x \mid x > p\}$. Since p is covered, we have $b \neq p$ and thus $p \lessdot b$. If $p < x_1, p < x_2$ then $b \leq x_1, b \leq x_2$ and hence $b \leq x_1 \wedge x_2 = p$, a contradiction. Therefore, p must be x_1 or x_2. ■

IV.3 Comparison

$$\begin{tikzcd}
 & & (\overline{T}_<)\&(\overline{S}_2) \arrow[dll, "{[a]}"'] \arrow[dr, "{[b]}"] & & \\
(\overline{T}_2) \equiv (\overline{T}_<) \arrow[d, "{[c]}"'] & & & (\overline{S}_2) \arrow[d, "{[d]}"] & \\
(T_2') \arrow[rrr, "{[f]}"] \arrow[d, "{[e]}"'] & & & (\text{DS-Haus}) \arrow[dl, "{[g]}"'] \arrow[dr, "{[h]}"] & \\
(S_<) \arrow[d, "{[i]}"'] & & (S_{ww}) & & (S_2') \\
(S) \arrow[r, "{[j]}"'] & (S_w) \arrow[ur, "{[k]}"] & & &
\end{tikzcd}$$

Figure D: Hausdorff type axioms

Proposition IV.3.1. *The implications* $[a] - [k]$ *in figure D hold.*

Proof. [a], [b], [c], [d]: trivial.

[e], [f]: from IV.1.1.

[g]: For the sake of contradiction, let p be semiprime yet not prime. Thus, there are $x_1, x_2 \neq p$ with $x_1 \wedge x_2 = p$, which means $x_1 \not\leq x_2$ and $x_1 \not\geq x_2$. Therefore, $x_1 \vee x_2 \notin \{x_1, x_2\}$, and using weakly Hausdorff property there exist

$u \not\leq x_1, v \not\leq x_2$ with $u \wedge v = 0$. Hence, $u, v \not\leq x_1 \wedge x_2 = p$, contradicting the semiprimeness of p.

[h]: trivial.

[i]: For the sake of contradiction, consider a semiprime p that is not a coatom, that is, $p < a < 1$ for some a. By $(S_<)$ we obtain $u \not\leq a, v \not\leq p$ such that $u \wedge v = 0$. Hence, using semiprimeness, $u \leq p$ and thus $u < a$, a contradiction.

[j], [k]: from IV.2.1. ∎

IV.4 Adding the subfitness

Subfitness cause $(\overline{S}_2), (S_2'), (\overline{T}_2), (T_2')$ and (DS-Haus) to coincide as it is seen from

Theorem IV.4.1. (S_2')&*(Sfit) implies* $(\overline{S}_2)$&$(T_<)$.

Lemma IV.4.2. (S_2')&*(Sfit) implies (DS-Haus).*

Proof. Let $a \vee b \notin \{a, b\}$. Firstly, $a \not\geq b$ means $a \neq 1$.

Secondly, since $a \not\leq b$, by subfitness we have some c with $a \vee c = 1 \neq b \vee c$. Thus, $a \not\leq b \vee c$ (otherwise $b \vee c \geq a \vee c = 1$) and again $b \vee c \neq 1$.

As also $a \vee (b \vee c) = 1$, by (S_2') we get $u \not\leq a$ and $v \not\leq b \vee c$ (leading to $v \not\leq b$) such that $u \wedge v = 0$. ∎

Lemma IV.4.3. *(DS-Haus)&(Sfit) implies* $(\overline{S}_2)$.

Proof. Let $a \not\leq b$ and $a \not\geq b$. By (Sfit) we have c_1, c_2 such that $a \vee c_1 = 1 \neq b \vee c_1$ and $a \vee c_2 \neq 1 = b \vee c_2$. Furthermore, if $b \vee c_1 \leq a \vee c_2$ then $b \leq a \vee c_2$, and consequently, $1 = b \vee c_2 \leq a \vee c_2$; thus $b \vee c_1 \not\leq a \vee c_2$ and symmetrically $b \vee c_1 \not\geq a \vee c_2$.

By (DS-Haus) we get $u' \not\leq a \vee c_2$ and $v' \not\leq b \vee c_1$ such that $u' \wedge v' = 0$.

Finally, set $u := u' \wedge b$ and $v := v' \wedge a$. We see that $u \not\leq a$ since otherwise $u' = u' \wedge 1 = u' \wedge (b \vee c_2) = (u' \wedge b) \vee (u' \wedge c_2) \leq a \vee c_2$. Likewise, $v \not\leq b$. ∎

Lemma IV.4.4. $(\overline{S}_2)$&*(Sfit) implies* $(T_<)$.

Proof. Let $1 \neq a > b$. Once more, by (Sfit) we have c with $a \vee c = 1 \neq b \vee c$. Because $a \vee (b \vee c) = 1 \notin \{a, b \vee c\}$, we can apply $(\overline{S}_2)$ to a and $b \vee c$, thus obtaining u, v such that $u \not\leq a, v \not\leq b \vee c$ (implying $v \not\leq b$), $v \leq a$ and $u \wedge v = 0$. ∎

Proof of **IV.4.1**. Use lemmata IV.4.2, IV.4.3 and IV.4.4. ∎

V. Regularity

V.1 The T_3-axiom

Definition (T_3)**.** A topological space X is *regular* (or satisfies the T_3-*axiom*) if

for any $x \in X$ and any closed A such that $A \not\ni x$ there are disjoint open V_1, V_2 such that $V_1 \ni x$ and $V_2 \supseteq A$.

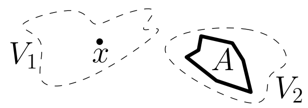

Figure E: T_3 property

Even though the definition concerns points, it may be reformulated without them.

Proposition V.1.1. *A space X is regular iff each $U \in \Omega(X)$ can be described as*

$$U = \bigcup\{V \in \Omega(X) \mid \overline{V} \subseteq U\}.$$

Proof. The inclusion $\supseteq$ is obvious.

$\Rightarrow$: Choose an $x \in U$. Since $A := X \setminus U \not\ni x$ is closed, the existence of V_1 and V_2 from (T_3) is stipulated. The disjointedness $V_1 \cap V_2 = \varnothing$ leads to $V_1 \subseteq X \setminus V_2$, which is a closed set. Therefore, $\overline{V_1} \subseteq X \setminus V_2$.

Furthermore, the relation $A \subseteq V_2$ is equivalent to the relation $X \setminus V_2 \subseteq X \setminus A = U$. Hence, for $V_x := V_1$ one has $\overline{V_x} \subseteq X \setminus V_2 \subseteq U$. Such a system $\{V_x \mid x \in U\}$ constitutes a subset cover of U, and consequently, completes the proof of the inclusion $\subseteq$.

$\Leftarrow$: With $U := X \setminus A$ take V_x from above and set $V_1 := V_x,\ V_2 := X \setminus \overline{V_x}$. ■

Now we will deal with the closure in the formula:

Lemma V.1.2. $\overline{V} \subseteq U \quad \equiv \quad \exists W \in \Omega(X), \quad W \cap V = \varnothing \ \&\ W \cup U = X.$

Proof. $\Rightarrow$: Take $W := X \setminus \overline{V}$.

$\Leftarrow$: Recall closure's definition. Suppose $z \in \overline{V} \setminus U$. Since $W \cup U = X$, the z must lie in the W; yet the W does not intersect the V. Thus, $z \notin \overline{V}$. ■

Without loss of generality, the W may be replaced by the pseudocomplement V^* in $\Omega(X)$, that is, by the open set $X \setminus \overline{V}$.

Remark V.1.3. Pseudocomplements always exist in frames: setting

$$a^* := \bigvee\{x \mid x \wedge a = 0\},$$

we have

$$a^* \wedge a = \bigvee\{x \mid x \wedge a = 0\} \wedge a = \bigvee\{x \wedge a \mid x \wedge a = 0\} = 0$$

(the second equality from the frame distributivity), and hence thus defined a^* is the largest x such that $a \wedge x = 0$.

V.2 Regular locales

Notation ($\prec$)**.**

$$V \prec U \quad \equiv \quad V^* \vee U = 1,$$

which is referred to by stating that *"V is rather below U"*.

Lemma V.2.1. *$a \prec b \Rightarrow a \leq b$.*

Proof. Using the distributivity we have

$$a = a \wedge 1 = a \wedge (a^* \vee b) = (a \wedge a^*) \vee (a \wedge b) = 0 \vee (a \wedge b) = a \wedge b. \qquad \blacksquare$$

With $\prec$ notation we are able to adopt the characterization V.1.1 by defining regularity as follows:

Definition (Reg)**.** A locale is called *regular* if

$$a = \bigvee\{x \mid x \prec a\}$$

for all its elements a.

Thus, we observe that

a topological space X is regular iff $\Omega(X)$ is regular.

V.3 Strength of regularity

Lemma V.3.1.

(i) $\bigvee_{i \in J}(a_i \oplus b) = \left(\bigvee_{i \in J} a_i\right) \oplus b$.

(ii) $\bigvee_{i \in J}(a \oplus b_i) = a \oplus \left(\bigvee_{i \in J} b_i\right)$.

(iii) $\left(\bigvee_{i \in J} a_i\right) \oplus \left(\bigvee_{j \in J} b_j\right) = \bigvee_{i,j \in J}(a_i \oplus b_j)$.

Proof. (i): Recall that $a \oplus b = \iota_1(a) \wedge \iota_2(b)$. We have

$$\left(\bigvee_{i\in J} a_i\right) \oplus b = \iota_1\left(\bigvee_{i\in J} a_i\right) \wedge \iota_2(b) = \left(\bigvee_{i\in J} \iota_1(a_i)\right) \wedge \iota_2(b)$$
$$= \bigvee_{i\in J} (\iota_1(a_i) \wedge \iota_2(b)) = \bigvee_{i\in J} (a_i \oplus b)$$

as frame homomorphisms ι_i preserve suprema (for the second equality) and as we have the frame distributivity (for the third equality).

(ii): Analogously by symmetry.

(iii): By consecutive application of (i) and (ii). ■

Lemma V.3.2. *For a general locale L and any of its saturated $U \supseteq d_L$ we have*

$$(a \wedge b, a \wedge b) \in U \qquad \Longrightarrow \qquad \forall x \prec a,\ y \prec b:\ (x, y) \in U$$

Proof. Beginning with $(x, y) \in x \oplus y$, we rewrite

$$\begin{aligned} x \oplus y &= (x \wedge (y^* \vee b)) \oplus (y \wedge (x^* \vee a)) \\ &= (\ (x \wedge y^*)\ \vee\ (x \wedge b)\ ) \oplus (\ (y \wedge x^*)\ \vee\ (y \wedge a)\ ) \end{aligned}$$

(using the "rather-belowness" and the distributivity, respectively).

Proceeding with (iii) of Lemma V.3.1 ,

$$\begin{aligned} \ldots &= (\ (x \wedge y^*) \oplus (y \wedge x^*)\ ) \vee (\ (x \wedge y^*) \oplus (y \wedge a)\ ) \vee \\ &\vee (\ (x \wedge b) \oplus (y \wedge x^*)\ ) \vee (\ (x \wedge b) \oplus (a \wedge y)\ ) \\ &\subseteq (y^* \oplus y) \vee (y^* \oplus y) \vee (x \oplus x^*) \vee (\ (a \wedge b) \oplus (a \wedge b)\ ) \end{aligned}$$

where the upper-bound of the last member follows from V.2.1 .

Besides, the ultimate expression is a subset of U: as $(x^* \oplus x), (y^* \oplus y) \subseteq d_L$ and $(a \wedge b) \oplus (a \wedge b) \subseteq U$ from the premise. In conclusion, $(x, y) \in x \oplus y \subseteq U$. ■

Theorem V.3.3. *(Reg) implies (I-Haus).*

Proof. We will show that regular locales satisfy (3.a) from III.3.5 (see page 15). Let $(a \wedge b, a \wedge b) \in U$. By the previous lemma, we have $(x, y) \in U$ for every $x \prec a$ and $y \prec b$. Recall the saturatedness on page 8 . Since U is saturated,

$$\{(x, y) \mid x \prec a\} \subseteq U \Longrightarrow \left(\bigvee\{x \mid x \prec a\}, y\right) \in U,$$

or equally, using regularity

$$(a, y) \in U$$

for all $y \prec b$. Therefore, in the same way: $(a, b) = (a, \bigvee\{y \mid y \prec b\}) \in U$. ■

By Theorem III.3.3 on page 15 , we obtain

Corollary V.3.4. *(Reg) implies (DS-Haus).*

It is useful to characterize (Reg) by a formula that resembles (Sfit):

Lemma V.3.5. *A locale is regular iff*

$$a \not\leq b \qquad \Rightarrow \qquad \exists c, \quad a \vee c = 1 \quad \& \quad c^* \not\leq b.$$

Proof. $\Rightarrow$: Suppose $a \not\leq b$; using regularity, there is $x \prec a$ with $x \not\leq b$ (otherwise $\bigvee\{x \mid x \prec a\} \leq b$). Let $c := x^*$. In other words, $x \prec a$ leads to $a \vee c = a \vee x^* = 1$. Additionally, $c^* \not\leq b$: or otherwise (from a standard pseudocomplement property) $x \leq x^{**} = c^* \leq b$.

$\Leftarrow$: By V.2.1 we always have the inequality $\bigvee\{x \mid x \prec a\} \leq a$. Hence, for the sake of contradiction, let us assume $a \not\leq \bigvee\{x \mid x \prec a\}$. Then there exists c from the premise. Specifically, $1 = a \vee c \leq a \vee c^{**}$, which gives us $c^* \prec a$; and yet $c^* \not\leq \bigvee\{x \mid x \prec a\}$. ■

Theorem V.3.6. *(Reg) implies (Sfit).*

Proof. Let $a \not\leq b$. From Lemma V.3.5 we get an element c such that $a \vee c = 1$ and $c^* \not\leq b$. What is more, it satisfies $b \vee c \neq 1$. If not so then by the distributivity

$$b = b \vee 0 = b \vee (c \wedge c^*) = (b \vee c) \wedge (b \vee c^*) = 1 \wedge (b \vee c^*) = b \vee c^*,$$

which contradicts $c^* \not\leq b$. ■

Remark V.3.7. By Theorem IV.4.1 on page 20 the regularity implies all Hausdorff type axioms mentioned in Chapter IV.

Proposition V.3.8. *Every sublocale S of a regular locale L is also regular.*

Proof. Let $h\colon L \to S$ be a frame homomorphism. By V.1.2 we can formulate "rather-belowness" as

$$a \prec b \;\equiv\; \exists c, \quad a \wedge c = 0 \text{ and } c \vee b = 1.$$

Since $h(a) \wedge h(c) = h(a \wedge c) = 0$ and $h(c) \vee h(b) = h(c \vee b) = 1$, we have $a \prec b \Rightarrow h(a) \prec h(b)$. Thus, $\{h(x) \mid x \prec a\} \subseteq \{y \mid y \prec h(a)\}$, and hence, $h(a) = h(\bigvee\{x \mid x \prec a\}) = \bigvee\{h(x) \mid x \prec a\} \leq \bigvee\{y \mid y \prec h(a)\}$. The other inequality is seen from V.2.1.

Because a sublocale homomorphism h is onto, for any $b \in S$ there is $a \in L$ such that $b = h(a) = \bigvee\{y \mid y \prec h(a)\} = \bigvee\{y \mid y \prec b\}$. ■

Remark V.3.9 (Fitness). Unlike the regularity, the subfitness is not hereditary: in general, not every sublocale of a subfit locale is subfit. The hereditary subfitness is called *fitness* and by Proposition V.3.8 and Theorem V.3.6 we have

Observation. (Reg) implies the fitness.

VI. Complete regularity

VI.1 The $T_{3\frac{1}{2}}$-axiom

Definition ($T_{3\frac{1}{2}}$)**.** A topological space X is said to be *completely regular* (or equally $T_{3\frac{1}{2}}$*-space*) if

for any $x \in X$ and any closed A such that $A \not\ni x$ there is a continuous function $\varphi\colon X \to \mathbb{I}$ with $\varphi(x) = 0$ and $\varphi[A] = \{1\}$.

Remark VI.1.1. The complete regularity implies regularity. (Take the continuous function φ from the definition and set $V_1 := \varphi^{-1}[\langle 0, \frac{1}{2}\langle]$ and $V_2 := \varphi^{-1}[\rangle \frac{1}{2}, 1\rangle]$.)

VI.2 Completely regular locales

Notation ($\prec\!\!\prec$)**.** For elements a, b of a locale L let us write

$$a \prec\!\!\prec b$$

and say *"a is completely below b"* if for every $r \in \langle 0, 1\rangle \cap \mathbb{Q}$ there exist $a_r \in L$ satisfying

$$\begin{aligned} &a_0 = a, \\ &a_1 = b, \\ &a_p \prec a_q \quad \text{for } p < q \end{aligned}$$

Remark VI.2.1. Clearly, the relation *interpolates*. On the other hand, for each interpolating subrelation $R \subseteq \prec$ and any aRb there are countably many elements lying (densily) between them.[1] Thus, $R \subseteq \prec\!\!\prec$, and therefore,

$\prec\!\!\prec$ is the largest interpolative subrelation of $\prec$.

Remark VI.2.2. By V.2.1 on page 22 $a \prec\!\!\prec b$ implies $a \leq b$.

Lemma VI.2.3. *Let X be a topological space and let $\{A_r \in \Omega(X) \mid r \in \langle 0, 1\rangle \cap \mathbb{Q}\}$ be a set witnessing $A \prec\!\!\prec B$. Then function $f\colon X \to \mathbb{I}$ defined by*

$$f(x) := \inf\{r \mid A_r \ni x\}$$

is continuous.

[1] for instance, by induction on dyadic fractions

Proof. Firstly,

$$x \in f^{-1}[\langle 0, s\langle] \;\equiv\; f(x) < s \;\equiv\; \exists r < s,\; A_r \ni x \;\equiv\; x \in \bigcup\{A_r \mid r < s\},$$

the second equivalence from a standard infimum property in linearly ordered sets. Secondly,

$$\begin{aligned} x \in f^{-1}[\rangle s, 1\rangle] \;&\equiv\; f(x) > s \\ &\equiv\; \exists r, r' \in \rangle s, f(x)\langle\,, \quad r' > r \text{ and } x \notin A_{r'} \text{ and } x \notin \overline{A_r} \\ &\equiv\; x \notin \bigcap\left\{\overline{A_r} \mid r > s\right\} \equiv\; x \in \mathbb{I} \setminus \bigcap\left\{\overline{A_r} \mid r > s\right\}, \end{aligned}$$

the latter equivalence by the density of rational numbers, the fact that infimum is an upper-bound and $A_r \prec A_{r'}$ (or equivalently $\overline{A_r} \subseteq A_{r'}$).

To sum up, preimages

$$\begin{aligned} f^{-1}[\langle 0, s\langle] &= \bigcup\{A_r \mid r < s\}, \\ f^{-1}[\rangle s, 1\rangle] &= \mathbb{I} \setminus \bigcap\left\{\overline{A_r} \mid r > s\right\} \end{aligned}$$

of $\mathbb{I}$'s subbasis are open; hence, f is continuous. ∎

Proposition VI.2.4. *A topological space X is completely regular iff*

$$\forall U \in \Omega(X)\colon \quad U = \bigcup\{V \in \Omega(X) \mid V \prec\!\!\prec U\}$$

Proof. $\Rightarrow$: We will show the inclusion $\subseteq$. Choose an arbitrary $x \in U$. Take continuous φ separating x from closed $A := X \setminus U$ and set

$$V_r = \begin{cases} \varphi^{-1}[\langle 0, \frac{r+1}{2}\langle] & \text{iff } r \in \langle 0, 1\langle \cap \mathbb{Q} \\ U & \text{iff } r = 1. \end{cases}$$

These open sets affirm $V_0 \prec\!\!\prec U$. Indeed: whenever $p < q$, we get

$$\overline{V_p} = \overline{\varphi^{-1}\left[\left\langle 0, \frac{p+1}{2}\right\langle\right]} \subseteq \varphi^{-1}\left[\left\langle 0, \frac{p+1}{2}\right\rangle\right] \subseteq \varphi^{-1}\left[\left\langle 0, \frac{q+1}{2}\right\langle\right] = V_q,$$

(the second inclusion since intervals $\langle 0, r\rangle$ are closed in $\mathbb{I}$). Further, $x \in V_0$ since $\varphi(x) = 0$. Such sets V_0 (for every $x \in U$) cover all of the U; hence, the inclusion $\subseteq$ holds.

$\Leftarrow$: For an open $U := X \setminus A$ and $x \in U$ there is a V with $x \in V \prec\!\!\prec U$. Thus, the f from VI.2.3 is the desired continuous function: first of all, $x \in V = V_0$ means $f(x) = 0$; furthermore, if $y \in A$ then $f(y) = \inf \varnothing = 1$. ∎

Definition (CReg)**.** A locale is called *completely regular* if

$$a = \bigvee\{x \mid x \prec\!\!\prec a\}$$

for all its elements a.

Besides, from Proposition VI.2.4 we gather that

a space X is completely regular iff $\Omega(X)$ is completely regular.

Remark VI.2.5. Because $\{x \mid x \prec\!\!\prec a\} \subseteq \{x \mid x \prec a\}$, (CReg) implies (Reg).

VII. Normality

VII.1 The T_4-axiom

Definition (T_4)**.** A topological space X is *normal* (or T_4-*space*) if

for every disjoint closed A, B there exist disjoint open V_1, V_2 with $V_1 \supseteq A$, $V_2 \supseteq B$.

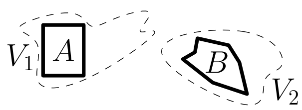

Figure F: T_4 property

VII.2 Normal locales

The point-free counterpart to (T_4) is straightforward:

Definition (Norm)**.** A locale L is *normal* if

$$a \vee b = 1 \qquad \Rightarrow \qquad \exists u, v, \quad u \wedge v = 0 \text{ and } a \vee u = 1 = b \vee v.$$

Because the definition of (T_4) is virtually point-free, we get that

a topology X is normal iff the locale $\Omega(X)$ is normal.

Remark VII.2.1. As $u \wedge v = 0 \ \equiv \ v \leq u^*$, we may define (Norm) equally by

$$a \vee b = 1 \qquad \Rightarrow \qquad \exists u, \quad a \vee u = 1 = b \vee u^*.$$

Lemma VII.2.2. *The relation $\prec$ interpolates in normal locales.*

Proof. Suppose a is rather below b. That is, $a^* \vee b = 1$, and using the normality,[1] we get $u \in L$ such that $a^* \vee u = 1 = b \vee u^*$. In other words, $a \prec u \prec b$. ■

Combined with Remark VI.2.1 on page 25 we form

Corollary VII.2.3. *(Norm) implies $\prec = \prec\!\!\prec$; hence, in case of normal locales, regularity coincides with complete regularity.*

Theorem VII.2.4. *(Norm) & (Sfit) $\Rightarrow$ (Reg).*

[1]more precisely, the subsequent remark

Proof. Take $a \in L$ and put $b := \bigvee\{x \mid x \prec a\}$. For the sake of contradiction, suppose that $a \not\leq b$. Using the subfitness, there is $c \in L$ with $a \vee c = 1 \neq b \vee c$. Applying the normality, we find $u \in L$ with $a \vee u^* = 1 = c \vee u$. Particularly, $u \prec a$, which leads to $u \leq b$. Finally, $b \vee c \geq u \vee c = 1$; contradicting the subfitness. ■

Since $\prec$ equals $\prec\!\!\prec$ in normal locales, we also deduce

Corollary VII.2.5. *(Norm)* & *(Sfit)* $\Rightarrow$ *(CReg)*.

Conclusion

Point-free topology plays, increasingly, important role in modern mathematics and computer science. Therefore, it is important to know which concepts and notions of point-set topology have natural counterparts in the point-free context.

In this thesis we observe that the relationships of T-axioms in classical and in pointless topology are alike: while for classical axioms it holds that

$$(T_4)\&(T_1) \implies (T_{3\frac{1}{2}})\&(T_1) \implies (T_3)\&(T_1) \implies (T_2) \implies (T_1) \implies (T_0),$$

for their point-free counterparts we have

$$\begin{aligned}(\text{Norm})\&(\text{Sfit}) \;\Rightarrow\; (\text{CReg}) \;\Rightarrow\; (\text{Reg}) \;\Rightarrow\; (\text{I-Haus})\&(\text{Sfit}) \;\Rightarrow \\ \Rightarrow\; (\text{DS-Haus})\&(\text{Sfit}) \;\Rightarrow\; \text{all axioms from chapter IV.}\end{aligned}$$

Bibliography

[1] Dowker, Clifford Hugh and Strauss, Dona Papert. Separation axioms for frames. In: *Topics in Topology.* 1972, vol. 8, pp. 223-240. Proc. Colloq., Keszthely (Hungary), 1972. Colloq. Math. Soc. Janos Bolyai, Amsterdam: North-Holland, 1974.

[2] Isbell, John Rolfe. Atomless parts of spaces. In: *Mathematica Scandinavica.* 1972, 31:5-32.

[3] Johnstone, Peter and Shu-Hao, Sun. Weak products and Hausdorff locales. Preprint.

[4] Paseka, Jan and Šmarda, Bohumil. T_2-frames and almost compact frames. In: *Czechoslovak Math. Journal.* 1992, vol. 42 (117), pp. 385-402. Praha: Czechoslovak Math. Journal, 1992.

[5] Picado, Jorge and Pultr, Aleš. *Frames and Locales: Topology without points.* 2012 edition. Basel: Birkhäuser, 2012. ISBN 978-3-0348-0153-9.

[6] Rosický, J. and Šmarda, B. T_1-locales. In: *Math. Proc. Cambr. Phil. Soc.* 1985, vol. 98, pp. 81-86.